Analyzing Music Using Group Theory

Rachel Wraley
Faculty Advisor: Dr. Wilson

Department of Mathematics
Butler University

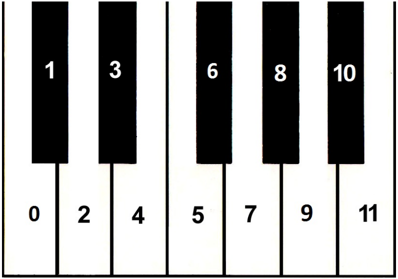
Senior Seminar, December 11, 2015

Outline

- Pitch is how high or low a note is, or the higher the frequency of a given note the higher the pitch and so on.
  In Music Theory the various different letter names (D, A$\flat$) are called Pitch Classes.
  For this talk we will use represent Pitch Classes using integers *mod* 12 with C=0.
- For example:
  C $\mapsto$ 0
  C$\sharp$ $\mapsto$ 1
  $\vdots$
  B $\mapsto$ 11

Intervals

- The distances between two notes or pitches. This distance is measured by half steps.
- For example:
  The distance from C♯(1) up to E(4) is an interval of 3.
- The distance from any E(4) up to the next C♯ is 9.



Triads

Definition

- A Triad is a "collection" of three notes that has specific spacing depending on whether or not the triad is a Major or Minor triad.
- Triads are what make up the chords we are familiar with in music.

Triads

Triad Spacing

A Triad is a set of 3 notes (n_1 the root, n_2 the third, and n_3 the fifth) that can be arranged to make either a Major or Minor Triad.

- Major Triads are created when arranged notes satisfy the following criteria:
  - $n_2 - n_1 = 4 \; mod(12)$
  - $n_3 - n_2 = 3 \; mod(12)$
- Minor Triads are created when arranged notes satisfy the following criteria:
  - $n_2 - n_1 = 3 \; mod(12)$
  - $n_3 - n_2 = 4 \; mod(12)$

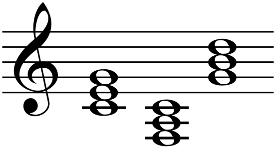


Figure: 3 examples of Major Triads that meet this criteria

Triads

Basics of Triad Transformations

- Triads can be written as an element of $\Delta = (\sigma, r) \in \mathbb{Z}_2 \times \mathbb{Z}_{12}$ where $\sigma \in \{+, -\}$ is the sign that distinguishes between Major(+) and Minor(-) and r is the root of the triad, written as a integer *mod* 12.
- Triads can undergo transformations such as transpositions to make a new triad.

Definition

A Transformation T is a bijective function
$T : \mathbb{Z}_2 \times \mathbb{Z}_{12} \to \mathbb{Z}_2 \times \mathbb{Z}_{12}$

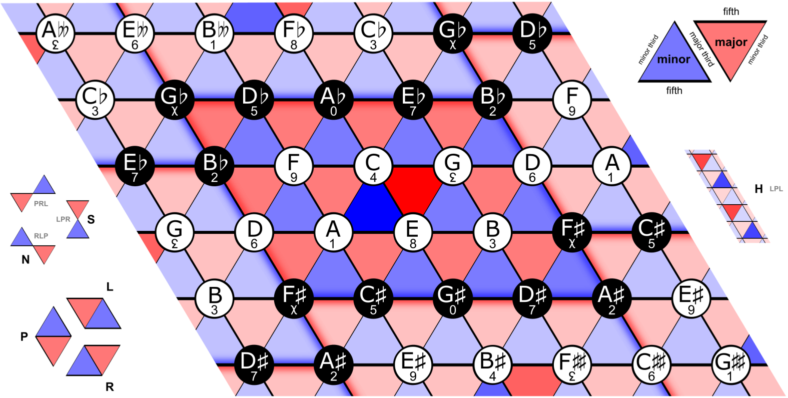
This means that $(\sigma, r) \mapsto (\sigma', r')$

Tonnetz

- Tonnetz: A conceptual lattice diagram representing traditional harmonic relationships between Major and Minor triads and scales.
- Music Theorist Hugo Riemann (Not to be confused to Bernhard Riemann of Riemann Sums Fame) created the Tonnetz picture on which there is a group structure of transformations acting on triads.
- Fun Fact: This diagram was first described in works by Leonhard Euler.

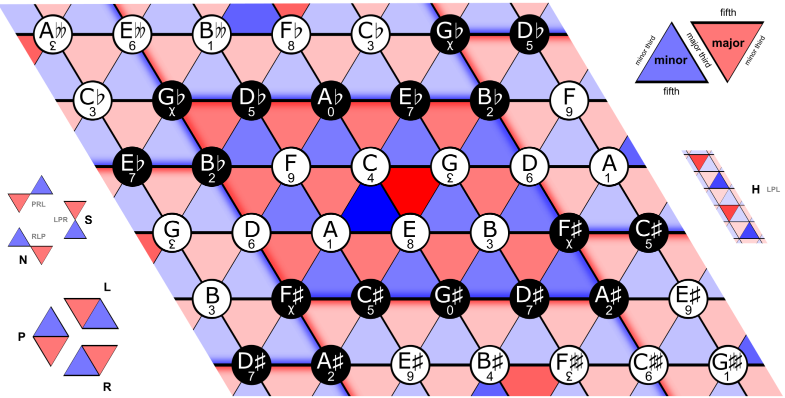
Triads

Riemann's Tonnetz



Triads

Riemann's Tonnetz

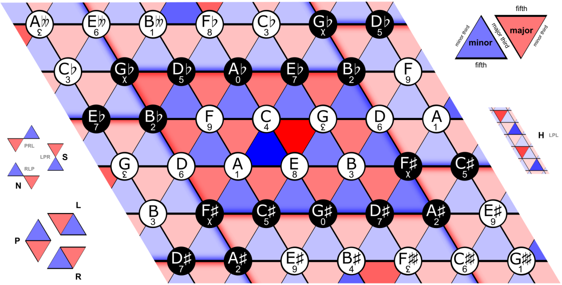


There are three main Riemannian Transformations that are "displayed" on Riemann's Tonnentz. There are Parallel, Relative, and Leading Tone Exchange. They act on the triangles by reflecting triangles across a certain triangle edge.

There are three main Riemannian Transformations that are "displayed" on Riemann's Tonnentz. There are Parallel, Relative, and Leading Tone Exchange.

- Parallel (P)
  - It takes a triad and changes its sign, called Mode Reversing.
  - Therefore Major Triad $\mapsto$ Minor triad and the reverse for Minor triads.
  - This is transformation reflects a triangle across the edge connecting the root and the fifth of a given triad.
- Relative (R)
  - It takes a Major(Minor) triad and finds its Relative Minor(Major) triad.
  - This is transformation reflects a triangle across the edge connecting the root and the third of a given triad.
- Leading Tone Exchange (L)
  - For a Major triad it moves the root down a semitone
  - For a Minor triad it moves the fifth up a semitone
  - This is transformation reflects a triangle across the edge connecting the third and the fifth of a given triad.

Riemannian Transformations
Examples

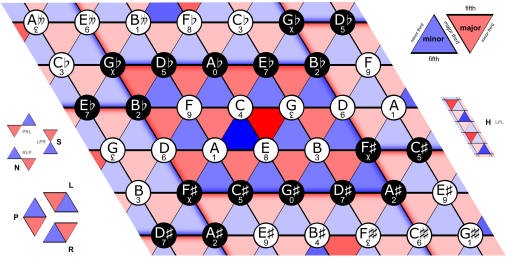


Example 1:

- R: $\overbrace{(9,-)}^{AMinor} \mapsto \overbrace{(0,+)}^{CMajor}$

$\overbrace{(2,-)}^{DMinor} \mapsto \overbrace{(5,+)}^{FMajor}$

$\overbrace{(0,+)}^{CMajor} \mapsto \overbrace{(9,-)}^{AMinor}$



Example 2:

- P: $(C,+) \mapsto (C,-)$
  $(D,+) \mapsto (D,-)$
  $(D,-) \mapsto (D,+)$

In fact, under R every triad gets transformed in this way:
The mode reverses with Major triads get transposed up by 9, and Minor triads get transposed up by 3.
We represent this by saying $R = \langle -, \underbrace{9}_{Major}, \underbrace{3}_{Minor} \rangle$.
Under P every triad gets transformed in this way:
The mode reverses with Major triads get transposed up by 0, and Minor triads get transposed up by 0.
We represent this by saying $P = \langle -, \underbrace{0}_{Major}, \underbrace{0}_{Minor} \rangle$.

Uniform Triadic Transformations (UTTs) (J. Hook, 2002)

- A triadic transformation T: $\mathbb{Z}_2 \times \mathbb{Z}_{12} \to \mathbb{Z}_2 \times \mathbb{Z}_{12}$ is *uniform* if it transforms all Major Triads "the same way" and Minor Triads in "the same way".
- The Notation of a UTT:
  $\langle \sigma, t^+, t^- \rangle$ where $\sigma \in \{+, -\}$ indicates whether the mode changes under the transformation and t^+ and t^- give the transposition change for Major and Minor triads respectively.
  Using the UTT notation, the definition works as follows:
  Given a Major Triad and a Mode Reversing UTT
    - $\Delta = (+, r)$ and T$= \langle -, t^+, t^- \rangle$
      $\Rightarrow T(\Delta) = T(+, r) = (-, r + t^+)$
      Given a Minor Triad and a Mode Reversing UTT
    - $\Delta = (-, r)$ and T $= \langle -, t^+, t^- \rangle$
      $\Rightarrow T(\Delta) = T(-, r) = (+, r + t^-)$

The UTT notation can now be used to represent the Riemannian Transformations P, R, and L as UTTs.

The UTT notation can now be used to represent the Riemannian Transformations P, R, and L as UTTs.

- $P = \langle -, 0, 0 \rangle$
- $R = \langle -, 9, 3 \rangle$
- $L = \langle -, 4, 8 \rangle$

The UTT notation can now be used to represent the Riemannian Transformations P, R, and L as UTTs.

- P $= \langle -, 0, 0 \rangle$
- R $= \langle -, 9, 3 \rangle$
- L $= \langle -, 4, 8 \rangle$

With this notation we would begin to answer questions like: What happens if one were to conjugate $T = \langle +, t^+, t^- \rangle$ by $P = \langle -, 0, 0 \rangle$?

Triads

Transformation Groups

There are specific Musical Groups that Music Theorists use to analyze musical pieces.

There are specific Musical Groups that Music Theorists use to analyze musical pieces.

Definition: Group of UTTs

$\mathcal{U} = \{\langle \sigma, t^+, t^- \rangle \mid \sigma \in \{+, -\}, t^+, t^- \in \mathbb{Z}_{12}\}$

There are specific Musical Groups that Music Theorists use to analyze musical pieces.

Definition: Group of UTTs

$\mathcal{U} = \{\langle \sigma, t^+, t^- \rangle \mid \sigma \in \{+, -\}, t^+, t^- \in \mathbb{Z}_{12}\}$

Definition

The Riemannian Group $\mathcal{R}$ is a group of functions generated by the Riemannian transformations P, R, and L. $\mathcal{R} \subseteq \mathcal{U}$

There are specific Musical Groups that Music Theorists use to analyze musical pieces.

Definition: Group of UTTs

$\mathcal{U} = \{\langle\sigma, t^+, t^-\rangle \mid \sigma \in \{+,-\}, t^+, t^- \in \mathbb{Z}_{12}\}$

Definition

The Riemannian Group $\mathcal{R}$ is a group of functions generated by the Riemannian transformations P, R, and L. $\mathcal{R} \subseteq \mathcal{U}$

Mini-Theorem

$\mathcal{R}$=$\{\langle\pm, n, 12-n\rangle \mid n \in \mathbb{Z}_{12}\}$

There are specific Musical Groups that Music Theorists use to analyze musical pieces.

Definition: Group of UTTs

$\mathcal{U} = \{\langle \sigma, t^+, t^- \rangle \mid \sigma \in \{+, -\}, t^+, t^- \in \mathbb{Z}_{12}\}$

Definition

The Riemannian Group $\mathcal{R}$ is a group of functions generated by the Riemannian transformations P, R, and L. $\mathcal{R} \subseteq \mathcal{U}$

Mini-Theorem

$\mathcal{R}$=$\{\langle \pm, n, 12 - n \rangle \mid n \in \mathbb{Z}_{12}\}$

Q: How does $\mathcal{U}$ relate to $\mathcal{R} = \{\langle -, t, 12 - t \rangle\}$ and to other subgroups of transformations?

Definition: Semi-Direct Product

Given a group homomorphism
φ: $\mathcal{K} \mapsto Aut(\mathcal{H})$
$k \mapsto [\varphi_k : \mathcal{H} \to \mathcal{H}]$,
defines the group $\mathcal{H} \rtimes \mathcal{K} = \{(h, k) | h \in \mathcal{H}, k \in \mathcal{K}\}$
with the rule $(h_1, k_1)(h_2, k_2) = (h_1 \cdot \varphi_{k_1}(h_2), k_1 k_2)$.

What happens if one were to Conjugate $\overbrace{\langle +, t^{+}, t^{-} \rangle}^{UTT}$ by $P = \langle -, 0, 0 \rangle$?

Uniform Triadic Transformations

UTT Examples

What happens if one were to Conjugate $\overbrace{\langle +, t^+, t^- \rangle}^{UTT}$ by $P = \langle -, 0, 0 \rangle$?

If $\Delta = (+, r)$:

$$(\langle -, 0, 0 \rangle \cdot \langle +, t^+, t^- \rangle \cdot \langle -, 0, 0 \rangle (\Delta)$$

$$(P \cdot \langle +, t^+, t^- \rangle \cdot P) \underbrace{(+, r)}_{\Delta}$$

$$= P\langle +, t^+, t^- \rangle(-, r)$$

$$= P(-, r + t^-)$$

$$= (+, r + t^-)$$

Similarly for Δ=$(-, r)$.
$(P \cdot \langle +, t^+, t^- \rangle \cdot P)(-, r) = (-, r + t^+)$

Similarly for Δ=$(-, r)$.
$(P \cdot \langle +, t^+, t^- \rangle \cdot P)(-, r) = (-, r + t^+)$
Major triads now get transposed by t^- and Minor triads by t^+.
$\therefore P\langle +, t^+, t^- \rangle P = \langle +, \underbrace{t^-}_{Major}, \underbrace{t^+}_{Minor} \rangle$.
Conjugation of $\langle +, t^+, t^- \rangle$ by P is the same as **swapping** $\langle t^+, t^- \rangle$ for $\langle t^-, t^+ \rangle \in \mathbb{Z}_{12} \times \mathbb{Z}_{12}$.

Group Theory

Musical Group Theory

Important fact: The Map

$\mathbb{Z}_2 \to Aut(\mathbb{Z}_{12} \times \mathbb{Z}_{12})$

0 (i.e "+") $\mapsto [Id : (\mathbb{Z}_{12} \times \mathbb{Z}_{12}) \to (\mathbb{Z}_{12} \times \mathbb{Z}_{12})]$

1 (i.e. "−") $\mapsto [Swap : (h_1, h_2) \mapsto (h_2, h_1)]$, is a homomorphism, so

We can construct $(\mathbb{Z}_{12} \times \mathbb{Z}_{12}) \rtimes \mathbb{Z}_2$.

Important fact: The Map
$\mathbb{Z}_2 \to Aut(\mathbb{Z}_{12} \times \mathbb{Z}_{12})$
0 (i.e "+") $\mapsto [Id : (\mathbb{Z}_{12} \times \mathbb{Z}_{12}) \to (\mathbb{Z}_{12} \times \mathbb{Z}_{12})]$
1 (i.e. "−") $\mapsto [Swap : (h_1, h_2) \mapsto (h_2, h_1)]$, is a homomorphism, so
We can construct $(\mathbb{Z}_{12} \times \mathbb{Z}_{12}) \rtimes \mathbb{Z}_2$.

Theorem (J. Hook):

$$\mathcal{U} = (\mathbb{Z}_{12} \times \mathbb{Z}_{12}) \rtimes \underbrace{\mathbb{Z}_2}_{\langle P \rangle}$$

Another Decomposition of $\mathcal{U}$.
Let $\mathcal{B}$=$\{\langle +, 0, t^{-}\rangle \mid t^{-} \in \mathbb{Z}_{12}\}$ be the group of Transpositions of only the Minor chords
So $\mathcal{B} \subseteq \mathcal{U}$.

Uniform Triadic Transformations

Musical Group Theory

Another Decomposition of $\mathcal{U}$.
Let $\mathcal{B}$=$\{\langle +, 0, t^{-}\rangle \mid t^{-} \in \mathbb{Z}_{12}\}$ be the group of Transpositions of only the Minor chords
So $\mathcal{B} \subseteq \mathcal{U}$.

Theorem (J. Hook)

$\mathcal{U}$=$\mathcal{R} \rtimes \mathcal{B}$

For Groups $\mathcal{U}, \mathcal{R}, \mathcal{B}$.
We can show $\mathcal{U} = \mathcal{R} \times \mathcal{B}$ as a set and $\mathcal{R}$ is normal in $\mathcal{U}$ so
(1) We have the following Map:
$\mathcal{B} \rightarrow Aut(\mathcal{R})$
$b \mapsto$ (conjugation by b) =[$\varphi_b : r \mapsto brb^{-1} \in \mathcal{R}$ ($\varphi_b \in Aut(\mathcal{R})$ because $\mathcal{R} \lhd \mathcal{U}$.
(2) Then $\underbrace{r_1 b_1}_{(r_1,b_1)} \cdot \underbrace{r_2 b_2}_{(r_2,b_2)} = (r_1 b_1 r_2 \cdot \underbrace{(b_1^{-1} b_1)}_{1} b_2 = \underbrace{r_1(b_1 r_2 b_1^{-1})}_{(r_1\varphi_{b_1}(r_2),} \underbrace{b_1 b_2}_{b_1 b_2)}$

Why is all this information important?

Why is all this information important?

- It is used by Music Theorists to analyze different pieces of music ranging from Mozart to modern tonal music.
- It also gives a language for analyzing how composers exploit relationships between keys and the interaction between them.

The End

Thank you.

Bibliography

Hook, Julian. Uniform Triadic Transformations. Journal of Music Theory 46.1/2 (2002): 57126. Web.