Analyzing Music with Group Theory

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1 Musical Background

Before we can discuss how Group Theory and Musical Analysis are interconnected, we need to discuss some basic music vocabulary.

1.1 Pitch, Pitch Class, and Intervals

Pitch is the used to define the highness or lowness of a note. The higher the frequency of a note the higher the pitch. For example the frequency of middle C is 261.2 Hz and the A above middle C is 440Hz. Additionally there are pitch classes that control how pitches and octaves are counted. In this paper we will use a Pitch Class that uses integers written *modulo* 12.
Pitch is how high or low a note is, or the higher the frequency of a given note the higher the pitch and so on.
In Music Theory the various different letter names (D, A$\flat$) are called Pitch Classes.
For this talk we will use represent Pitch Classes using integers *mod* 12 with C=0.
For example:
C $\mapsto$ 0
C$\sharp$ $\mapsto$ 1
$\vdots$
B $\mapsto$ 11

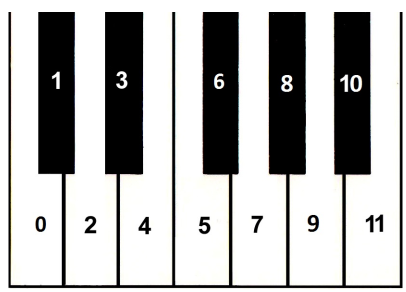

Figure 1: *Each Key with its Pitch Class value*

We use Intervals to measure the distances between to pitches. THat distance is measured in half steps, where a half step is the distance from one note to the next closest note. For example a half step is the distance between C(0) and C($\sharp$).

2 Triads

This sections focuses on Triads, Riemann's Tonnetz, and the basics of Triadic Transformations.
A Triad is a set of 3 notes (n_1 the root, n_2 the third, and n_3 the fifth) that can be arranged to make either a Major or Minor Triad. Major Triads are created when arranged notes satisfy the following criteria: $n_2 - n_1 = 4 \; mod(12)$ and $n_3 - n_2 = 3 \; mod(12)$. Minor Triads are created when arranged notes satisfy the following criteria: $n_2 - n_1 = 3 \; mod(12)$ and $n_3 - n_2 = 4 \; mod(12)$.

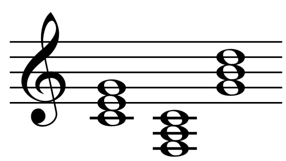

Figure 2: 3 examples of Major Triads that meet the Triad criteria

2.1 Triadic Transformations

Triads can be written as an element of $\Delta = (\sigma, r) \in \mathbb{Z}_2 \times \mathbb{Z}_{12}$ where $\sigma \in \{+, -\}$ is the sign that distinguishes between Major(+) and Minor(-) and r is the root of the triad (n_1 from earlier), written as a integer *mod* 12.

Triads can undergo transformations such as transpositions to make a new triad.

Definition 2.1. *A Transformation T is a bijective function*
T: $\mathcal{Z}_2 \times \mathbb{Z}_{12} \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_{12}$

This means that $\Delta \mapsto \Delta^{'}$

2.2 Riemann's Tonnetz

Riemann's Tonnetz is a conceptual lattice diagram representing traditional harmonic relationships between Major and Minor triads. It was first created by Bernhard Euler in a his 1739 *Tentamen novae theoriae musicae ex certissismis harmoniae principiis dilucide expositae.* In Music Theorist Hugo Riemann (Not to be confused to Bernhard Riemann of Riemann Sums Fame) used his version of Euler's Tonnetz (see figure 3) to show the group structure of transformations acting on triads.

There are three main Riemannian Transformations that are displayed on Riemann's Tonnentz. There are Parallel, Relative, and Leading Tone Exchange. They act on the triangles by reflecting triangles across a certain triangle edge. Parallel (P) takes a triad and changes its sign, called Mode Reversing. Therefore

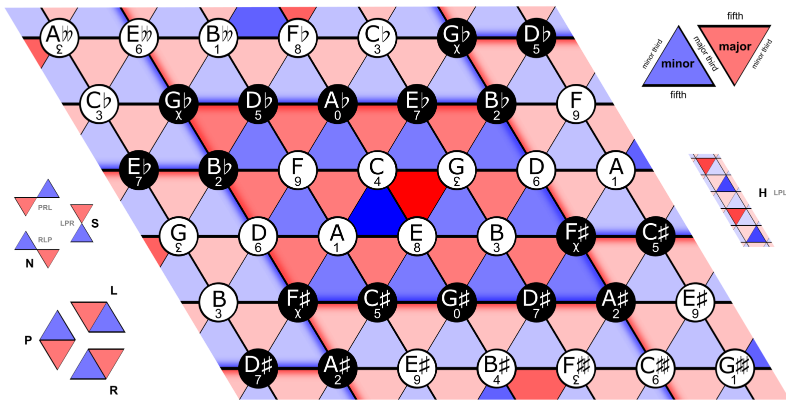

Figure 3: Riemann's Tonnetz

Major Triad $\mapsto$ Minor triad and the reverse for Minor triads. This is transformation reflects a triangle across the edge connecting the root and the fifth of a given triad.
Relative (R) takes a Major(Minor) triad and finds its Relative Minor(Major) triad. This is transformation reflects a triangle across the edge connecting the root and the third of a given triad.
Leading Tone Exchange (L) changes Major and Minor Triads differently. For a Major triad it moves the root (n_1) down a semitone. For a Minor triad it moves the fifth (n_3) up a semitone. This is transformation reflects a triangle across the edge connecting the third and the fifth of a given triad.

Example 2.1:

R: $\overbrace{(-,9)}^{AMinor} \mapsto \overbrace{(+,0)}^{CMajor}, \quad \overbrace{(-,2)}^{DMinor} \mapsto \overbrace{(+,5)}^{FMajor}, \quad \overbrace{(+,0)}^{CMajor} \mapsto \overbrace{(-,9)}^{AMinor}$

Example 2.2:

P: $\overbrace{(+,0)}^{CMajor} \mapsto \overbrace{(-,0)}^{CMinor}, \quad \overbrace{(+,2)}^{DMajor} \mapsto \overbrace{(-,2)}^{DMinor}, \quad \overbrace{(-,2)}^{DMinor} \mapsto \overbrace{(+,2)}^{DMajor}$

3 Uniform Triadic Transformations

3.1 Uniform Triadic Transformations (UTTs) (J. Hook, 2002)

A triadic transformation T: $\mathbb{Z}_2 \times \mathbb{Z}_{12} \to \mathbb{Z}_2 \times \mathbb{Z}_{12}$ is *uniform* if it transforms all Major Triads "the same way" and Minor Triads in "the same way".
The Notation of a UTT:

$\langle \sigma, t^+, t^- \rangle$ where $\sigma \in \{+,-\}$ indicates whether the mode changes under the transformation and t^+ and t^- give the transposition change for Major and Minor triads respectively.
Using the UTT notation, the definition works as follows:
Given a Major Triad and a Mode Reversing UTT: $\Delta =(+, r)$ and T=$\langle -, t^+, t^- \rangle$
$\Rightarrow T(\Delta) = T(+,r) = (-, r+t^+)$
Given a Minor Triad and a Mode Reversing UTT: $\Delta = (-, r)$ and T =$\langle -, t^+, t^- \rangle$
$\Rightarrow T(\Delta) = T(-,r) = (+, r+t^-)$
The UTT notation can now be used to represent the Riemannian Transformations P, R, and L as UTTs.
P = $\langle -, 0, 0 \rangle$
R = $\langle -, 9, 3 \rangle$
L = $\langle -, 4, 8 \rangle$

Example 3.1:

What happens if one were to Conjugate $\overbrace{\langle +, t^+, t^- \rangle}^{UTT}$ by $P = \langle -, 0, 0 \rangle$?
If $\Delta = (+, r)$:
$(\langle -, 0, 0 \rangle \cdot \langle +, t^+, t^- \rangle \cdot \langle -, 0, 0 \rangle(\Delta)$
$(P \cdot \langle +, t^+, t^- \rangle \cdot P) \underbrace{(+, r)}_{\Delta}$
$= P\langle +, t^+, t^- \rangle(-, r)$
$= P(-, r+t^-)$
$= (+, r+t^-)$

Similarly for Δ=$(-, r)$.
$(P \cdot \langle +, t^+, t^- \rangle \cdot P)(-, r) = (-, r+t^+)$
Major triads now get transposed by t^- and Minor triads by t^+.
$\therefore P\langle +, t^+, t^- \rangle P = \langle +, \underbrace{t^-}_{Major}, \underbrace{t^+}_{Minor} \rangle$.
Therefore Conjugation of $\langle +, t^+, t^- \rangle$ by P is the same as **swapping** $\langle t^+, t^- \rangle$ for $\langle t^-, t^+ \rangle \in \mathbb{Z}_{12} \times \mathbb{Z}_{12}$.

3.2 Musical Groups

There are specific Musical Groups that Music Theorists use to analyze musical pieces.

Definition 3.1. *Group of UTTs:* $\mathcal{U} = \{\langle \sigma, t^+, t^- \rangle \mid \sigma \in \{+,-\}, t^+, t^- \in \mathbb{Z}_{12}\}$

Definition 3.2. *Riemannian Group: The Riemannian Group $\mathcal{R}$ is a group of functions generated by the Riemannian transformations P, R, and L, and* $\mathcal{R} \subseteq \mathcal{U}$

Using these definition, we can begin to think about questions such as, How does $\mathcal{U}$ relate to $\mathcal{R}$ and to other Musical subgroups of transformations?

4 Group Theory

4.1 Semi-Direct Products

This section shows the Group Theory behind all of the Musical Groups and Subgroups.

Definition 4.1. *Semi-Direct Product Given a group homomorphism*
φ*:* $\mathcal{K} \mapsto Aut(\mathcal{H})$
$k \mapsto [\varphi_k : \mathcal{H} \to \mathcal{H}]$*,*
defines the group $\mathcal{H} \rtimes \mathcal{K} = \{(h,k) | h \in \mathcal{H}, k \in \mathcal{K}\}$
with the rule $(h_1, k_1)(h_2, k_2) = (h_1 \cdot \varphi_{k_1}(h_2), k_1 k_2)$*.*

4.2 Musical Group Theory

Important fact: The Map
$\mathbb{Z}_2 \to Aut(\mathbb{Z}_{12} \times \mathbb{Z}_{12})$
0 (i.e "+") $\mapsto [Id : (\mathbb{Z}_{12} \times \mathbb{Z}_{12}) \to (\mathbb{Z}_{12} \times \mathbb{Z}_{12})]$
1 (i.e. "−") $\mapsto [Swap : (h_1, h_2) \mapsto (h_2, h_1)]$, is a homomorphism, so we can construct $(\mathbb{Z}_{12} \times \mathbb{Z}_{12}) \rtimes \mathbb{Z}_2$.

Theorem 4.1. *(J. Hook,2002):* $\mathcal{U} = (\mathbb{Z}_{12} \times \mathbb{Z}_{12}) \rtimes \underbrace{\mathbb{Z}_2}_{\langle P \rangle}$

Another Decomposition of $\mathcal{U}$.
Let $\mathcal{B}$=$\{\langle +, 0, t^- \rangle \mid t^- \in \mathbb{Z}_{12}\}$ be the group of Transpositions of only the Minor chords and $\mathcal{B} \subseteq \mathcal{U}$.

Theorem 4.2. *(J. Hook,2002)* $\mathcal{U} = \mathbb{R} \rtimes \mathcal{B}$

Proof. For Groups $\mathcal{U}, \mathcal{R}, \mathcal{B}$. We can show $\mathcal{U} = \mathcal{R} \times \mathcal{B}$ as a set and $\mathcal{R}$ is normal in $\mathcal{U}$ so (1) We have the following Map: $\mathcal{B} \to Aut(\mathcal{R})$ where each element of $]\mathcal{B}$ works like this: $b \mapsto$ (conjugation by b) =$[\varphi_b : r \mapsto brb^{-1} \in \mathcal{R}$ $(\varphi_b \in Aut(\mathcal{R})$ because $\mathcal{R} \lhd \mathcal{U}$. (2) Then $(r_1 b_1) \cdot (r_2 b_2) = (r_1 b_1 r_2) \cdot (b_1^{-1} b_1) = (r_1(b_1 r_2 b_1^{-1}), b_1 b_2)$ ■

5 Why is it Important?

Musical Group Theory and plain Mathematical Group Theory has some serious application in the field of Music. It is used by Music Theorists to analyze different pieces of music ranging from Mozart to modern tonal music. It also gives Music Theorist a language for analyzing how composers exploit relationships between keys and the interaction between them.

References

[1] Julian Hook. *Uniform Triadic Transformations*. Journal of Music Theory, 46.1/2 (2002): 57-126. Web.